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From the Author.

ON

RELATED CAUSTICS OF REFLECTION

AND

THE EVOLUTE OF THE LEMNISCATA,

AS DERIVED FROM

A CAUSTIC OF THE HYPERBOLA.

George Frederick

BY THE REV. G. F. CHILDE, M.A. of Ch. Ch. Oxford

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ADDENDUM.

Page 5, § (v.), corollary.

The theorem contained in this article being written in the form,

$$ds = \frac{2}{3}\left(ds_x \frac{dx}{ds} + ds_y \frac{dy}{ds}\right)$$

expresses the geometrical relation, that an elementary arc of any reflecting curve is always equal to $\frac{2}{3}$ of the projections upon its tangent of the corresponding elementary arcs in the related caustics.

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RELATED CAUSTICS.

On some general properties of co-related caustic curves, generated by parallel incident rays; and on the equation to the evolute of the Lemniscata of Bernoulli, derived from the caustic of the Hyperbola.

(I.) In these pages it is designed to investigate some general relations, connecting caustics due to parallel incident rays with the evolute of the reflector, as being properties of caustic reflected curves which are certainly singular, and which, it is probable, may not have been previously noticed. It will be further shewn how the evolute of a given curve can be analytically represented by regarding it in its relation to a caustic of reflection, in an instance in which the direct investigation of its equation would be attended by difficulties of elimination possibly insuperable, but which may be in this manner evaded.

(II.) **Theorem.** If two caustics are generated simultaneously by rays reflected from any plane curve, $f(xy) = 0$, the incident rays being respectively parallel to the axes of x and y, *i.e.* to any two directions whatever, meeting each other at right angles; and if R is the radius of curvature of the reflector, R_x and R_y those of the two caustics, these three lines are connected by the equation,

$$R = \frac{4}{3}\left(R_x \frac{dx}{ds} + R_y \frac{dy}{ds}\right)$$

Let ξ_x η_x be co-ordinates of the caustic generated by rays incident parallel to the axis of x, ξ_y η_y those of the caustic which is touched by rays incident parallel to the axis of y; and assume, for the sake of convenience,

$$u = \frac{dy}{dx}; \; v = \frac{d^2y}{dx^2}; \; w = \frac{d^3y}{dx^3};$$

then it may be shewn that,

$$\xi_x = x + \frac{u}{2v}(1-u^2);\ \eta_x = y + \frac{u^2}{v};$$

$$\xi_y = x - \frac{u}{v};\ \eta_y = y + \frac{1}{2v}(1 - u^2);$$

and, after the differentiation of these expressions, we find,

$$\frac{d\eta_x}{d\xi_x} = \frac{2u}{1-u^2};\ \frac{d^2\eta_x}{d\xi_x{}^2} = \frac{4v^3}{(1-u^2)^3}\,\frac{(1+u^2)}{(3v^2-uw)};$$

$$\frac{d\eta_y}{d\xi_y} = -\frac{1-u^2}{2u};\ \frac{d^2\eta_y}{d\xi_y{}^2} = \frac{v^3(1+u^2)}{2u^3w};$$

consequently,

$$R_x = -\frac{3v^2 - uw}{4\,v^3}(1 + u^2)^2.$$

$$R_y = -\frac{w}{4v^3}(1 + u^2)^2.$$

Now, $$R_x + u\,R_y = -\frac{3\,(1 + u^2)^2.}{4v}$$

and, $$R = -\frac{(1 + u^2)^{\frac{3}{2}}}{v}$$

$$\therefore\ -\frac{(1 + u^2)^2}{v} = R\,(1 + u^2)^{\frac{1}{2}} = R\,\frac{ds}{dx}$$

$$\therefore\ R_x + R_y\frac{dy}{dx} = \frac{3}{4}\,R\,\frac{ds}{dx}$$

$$\therefore\ R = \frac{4}{3}\,(R_x\frac{dx}{ds} + R_y\frac{dy}{ds}): \qquad (1)$$

which is the theorem enunciated in this article.

Cor. The geometrical interpretation of the equation (1) is, that the radius of curvature of the reflector is always equal to $\frac{4}{3}$ of the sum of the projections of the radii of curvature of the two caustics upon the normal of the reflector.

In the figure (1), let Q_x and Q_y be corresponding points in the caustics, and let their radii of curvature meet the normal PG in the points L and M, then,

$$PLQ_x = \frac{\pi}{2} - LPQ_x = \frac{\pi}{2} - G = t.$$

$$PMQ_y = \frac{\pi}{2} - MPQ_y = \frac{\pi}{2} - MPS_y = G = \frac{\pi}{2} - t.$$

$$\therefore \cos PLQ_x = \cos t = \frac{dx}{ds};$$

$$\cos PMQ_y = \sin t = \frac{dy}{ds};$$

consequently, $$R = \tfrac{4}{3} \left\{ R_x \cos PLQ_x + R_y \cos PMQ_y \right\}.$$

(III.) THEOREM. If two caustics are generated by systems of parallel rays intersecting at right angles, as in section (II.), and if K is the distance between correspondent points in the caustics, R the radius of curvature of the reflector, there is always the relation,

$$R = 2K.$$

Let l_x or l_y be the distance between the reflector and its caustic, when rays are incident parallel to the axis of x or y; then we shall have,

$$l_x{}^2 = (x - \xi_x)^2 + (y - \eta_x)^2.$$

$$l_y{}^2 = (x - \xi_y)^2 + (y - \eta_y)^2.$$

and, after reducing these expressions by the formulæ in (II.), we find,

$$l_x = \frac{u}{2v}(1 + u^2); \qquad l_y = \frac{1}{2v}(1 + u^2).$$

Now, if ξ and η are corresponding co-ordinates to the evolute of the reflector,

$$\xi = x - \frac{u}{v}(1 + u^2); \quad \eta = y + \frac{1}{v}(1 + u^2);$$

$$\therefore \quad \xi = x - 2l_x \; ; \qquad\qquad \eta = y + 2l_y.$$

But,

$$R^2 = (x - \xi)^2 + (y - \eta)^2.$$

$$\therefore \quad R^2 = 4\,(l_x^2 + l_y^2).$$

It is geometrically evident that the systems of rays simultaneously reflected, intersect at right angles in the point of incidence,

$$\therefore \; l_x^2 + l_y^2 = Q_x\,Q_y^2 = K^2.$$

$$\therefore \quad R = 2\,K.$$

Hence, in every curve, the radius of curvature is equal to twice the distance between correspondent points in the co-related caustics.

Cor. In the circle, R is constant, and $= a$, the radius of the reflector; consequently, in this case of reflection, the distance between correspondent points in the two caustic epicycloids is constant, and is always equal to half the radius of the reflector, which is the distance between the cusps of either caustic, and the vertices of the other.

(IV.) **Theorem.** If k_x and k_y are the distances between correspondent points in the evolute of the reflector and the two caustics,

$$R^2 = \frac{4}{5}\,(k_x^2 + k_y^2).$$

Employing the formulæ of (II.), we shall find,

$$k_x^2 = (\eta - \eta_x)^2 + (\xi - \xi_x)^2 = \frac{1}{4v^2}\,(u^2 + 1)^2\,(u^2 + 4);$$

$$k_y^2 = (\eta - \eta_y)^2 + (\xi - \xi_y)^2 = \frac{1}{4v^2}\,(u^2 + 1)^2\,(4\,u^2 + 1);$$

or,

$$k_x = \frac{R}{2}\sqrt{\frac{u^2 + 4}{u^2 + 1}}\,; \qquad k_y = \frac{R}{2}\sqrt{\frac{4\,u^2 + 1}{u^2 + 1}}$$

Hence,

$$k_x^2 + k_y^2 = \frac{R^2}{4}\,\frac{5\,(u^2+1)}{u^2+1} = \frac{5}{4}R^2$$

$$\therefore\quad R^2 = \frac{4}{5}\,(k_x^2 + k_y^2).$$

(V.) THEOREM. If ds_x and ds_y are increments of the arcs of the related caustics, corresponding to simultaneous points in those curves; ds the corresponding increment in the arc of the reflector; the arcs of the three curves are connected by the relation,

$$ds^2 = \frac{2}{3}\,(ds_x\,dx + ds_y\,dy)$$

We have

$$\frac{d\eta_x}{d\xi_x} = \frac{2u}{1-u^2}$$

$$\therefore\quad 1 + \left(\frac{d\eta_x}{d\xi_x}\right)^2 = \left(\frac{1+u^2}{1-u^2}\right)^2 = \left(\frac{ds_x}{d\xi_x}\right)^2,$$

$$\therefore\quad \frac{ds_x}{d\xi_x} = \frac{1+u^2}{1-u^2} = \text{sec. } 2\,t.$$

$$\therefore\ \frac{ds_x}{dx} = \frac{1+u^2}{1-u^2}\,\frac{d\xi_x}{dx} = \frac{1+u^2}{1-u^2}\,\frac{1-u^2}{2\,v^2}\,(3v^2 - uw);$$

$$\therefore\quad \frac{ds_x}{dx} = \frac{1+u^2}{2v^2}\,(3v^2 - uw) \qquad (1)$$

Again,
$$\frac{d\eta_y}{d\xi_y} = -\,\frac{1-u^2}{2u}$$

$$\therefore\quad 1 + \left(\frac{d\eta_y}{d\xi_y}\right)^2 = \left(\frac{1+u^2}{2u}\right)^2 = \left(\frac{ds_y}{d\xi_y}\right)^2,$$

$$\therefore\quad \frac{ds_y}{d\xi_y} = \frac{1+u^2}{2u} = \text{cosec. } 2\,t.$$

And,

$$\frac{ds_y}{dx} = \frac{1+u^2}{2u}\,\frac{d\xi_y}{dx} = \frac{1+u^2}{2u}\,\frac{u\,w}{v^2}$$

$$\therefore \quad \frac{ds_{\text{y}}}{dx} = \frac{w}{2v^2}(1 + u^2) \qquad (2)$$

There is, also,

$$\frac{ds}{dx} = (1 + u^2)^{\frac{1}{2}}.$$

Hence we have,

$$\frac{ds_{\text{x}}}{dx} = \frac{1 + u^2}{2v^2}(3v^2 - uw)$$

$$= \frac{3}{2}(1 + u^2) - \frac{uw}{2v^2}(1 + u^2)$$

$$= \frac{3}{2}\left(\frac{ds}{dx}\right)^2 - u\frac{ds_{\text{y}}}{dx}$$

$$\therefore \quad \frac{3}{2}\left(\frac{ds}{dx}\right)^2 = \frac{ds_{\text{x}}}{dx} + \frac{dy}{dx}\frac{ds_{\text{y}}}{dx};$$

Consequently,

$$ds^2 = \frac{2}{3}(ds_{\text{x}}\,dx + ds_{\text{y}}\,dy);$$

which is the Theorem proposed: and it may be otherwise expressed under the form,

$$\frac{ds}{dx} = \frac{2}{3}\left(\frac{ds_{\text{x}}}{dx}\frac{dx}{ds} + \frac{ds_{\text{y}}}{dx}\frac{dy}{ds}\right).$$

(VI.) Let E be the angle included between the two lines k_{x} and k_{y} at their point of intersection on the evolute, then we may shew that,

$$\tan \text{E} = \frac{3}{4}\sin 2t;$$

t being the inclination of the tangent of the reflector to the axis of x.

In the triangle Q_x E Q_y, fig. 1., we have,

$$\text{Q}_x\,\text{Q}_y{}^2 = k_{\text{x}}{}^2 + k_y{}^2 - 2k_{\text{x}}\,k_y \cos \text{E}.$$

$$\therefore \quad \frac{R^2}{4} = \frac{5\,R^2}{4} - 2k_x\,k_y \cos E.$$

$$\therefore \quad \cos E = \frac{R^2}{2k_x\,k_y}.$$

After substituting for k_x and k_y we find from this,

$$\cos E = \frac{2\,(u^2 + 1)}{(u^2 + 4)^{\frac{1}{2}}\,(4u^2 + 1)^{\frac{1}{2}}}$$

Consequently,

$$\tan E = \frac{3u}{2\,(u^2 + 1)} = \frac{3\tan t}{2\sec^2 t} = \frac{3}{2}\sin t \cos t.$$

$$\therefore \quad \tan E = \frac{3}{4}\sin 2t.$$

(VII.) If A is the area of the triangle $Q_x\;E\;Q_y$
$A_{,}$ . . . . $Q_x\;P\;Q_y$
we shall find,

$$A = \frac{1}{2}\,k_x\,k_y \sin E = \frac{3}{8}\,\frac{R^2\,u}{u^2 + 1} = \frac{3}{16}\,R^2 \sin 2t = \left(\frac{R}{2}\right)^2 \tan E.$$

$$A_{,} = \frac{1}{2}\,l_x\,l_y = \frac{1}{8}\,\frac{R^2\,u}{u^2 + 1} = \frac{1}{16}\,R^2 \sin 2t = \frac{1}{3}\left(\frac{R}{2}\right)^2 \tan E.$$

$$\therefore \quad A = 3\,A_{,}.$$

Cor. The area of the quadrilateral figure $P\;Q_x\;E\;Q_y$ is equal to 4 times the area of the triangle $P\;Q_x\;Q_y$.

(VIII.) Ex. 1.—As an application of the Theorem in (V.) we take the case of the logarithmic curve.

The equation of the primitive is $x = \log y$; consequently, $u = v = w = y$, and the formulæ for the arcs become,

$$\frac{ds_x}{dx} = 1 + y^2; \quad \frac{ds_y}{dx} = \frac{1 + y^2}{2y}; \quad \frac{ds}{dx} = (1 + y^2)^{\frac{1}{2}};$$

Then,

$$ds_x\,dx + ds_y\,dy = (1 + y^2)\,dx^2 + \frac{1 + y^2}{2y}\,dx\,dy$$

$$= \frac{3}{2}(1 + y^2)\,dx^2 = \frac{3}{2}\,ds^2;$$

$$\therefore\ ds^2 = \frac{2}{3}(ds_x\,dx + ds_y\,dy).$$

Ex. 2.—To illustrate the Theorem in (II.) connecting the radii of curvature, let the reflector be the parabolic curve, $a^2 y = x^3$; then,

$$u = \frac{3x^2}{a^2};\quad v = \frac{6x}{a^2};\quad w = \frac{6}{a^2};$$

$$\frac{dx}{ds} = \frac{a^2}{(a^4 + 9x^4)^{\frac{1}{2}}};\quad \frac{dy}{ds} = \frac{3x^2}{(a^4 + 9x^4)^{\frac{1}{2}}};$$

hence we shall obtain,

$$\mathrm{R}_x = -\frac{5}{48}\frac{(a^4 + 9x^4)^2}{a^6 x};\quad \mathrm{R}_y = -\frac{(a^4 + 9x^4)^2}{144\,a^4\,x^3};$$

$$\mathrm{R} = -\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^4 x}.$$

$$\therefore\ \mathrm{R}_x\frac{dx}{ds} + \mathrm{R}_y\frac{dy}{ds} = -\frac{(a^4 + 9x^4)^{\frac{3}{2}}}{8a^4 x} = \frac{3}{4}\mathrm{R}.$$

Ex. 3.—In the common parabola, $y^2 = 4ax$, and we find,

$$u = \frac{2a}{y};\quad v = -\frac{4a^2}{y^3};\quad w = \frac{24a^3}{y^5}.$$

Hence,

$$\mathrm{R}_x = 0;\quad \mathrm{R}_y = \frac{3}{2}\frac{(x + a)^2}{a};$$

Also,

$$ds_x = 0;\quad ds_y = \frac{3}{2}\frac{x + a}{(ax)^{\frac{1}{2}}}\,dx;$$

$$R = \frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}; \quad ds = \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx;$$

from which expressions the general formulæ may be verified. The condition $R_x = 0$, indicates the property of parabolic reflection to the focus; and that this, as is well known, is the only curve to which this property belongs, may be readily shewn from the general values of the caustic radii of curvature.

1st. If $R_x = 0$, we shall have, page 2.,

$3v^2 - uw = 0$; or $v = \infty$; the latter of which cannot generally be true: we have, therefore, to integrate the former equation. The integral is,

$$y = 2(ax+b)^{\frac{1}{2}} + c$$

or,

$$(y-c)^2 = 4(ax+b);$$

which represents a parabola, whose axis coincides with, or is parallel to, the axis of x.

2nd. If $R_y = 0$, there will be the condition $w = 0$; of which the integral is,

$$y = ax^2 + bx + c;$$

representing a parabola whose axis coincides with, or is parallel to, the axis of y; $a\ b\ c$ being in each case the arbitrary constants.

Ex. 4. In the cycloid,

$$u = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}}; \quad v = -\frac{a}{x^{\frac{3}{2}}(2a-x)^{\frac{1}{2}}}; \quad w = -\frac{a(2x-3a)}{x^{\frac{5}{2}}(2a-x)^{\frac{3}{2}}}.$$

$$\frac{dx}{ds} = \left(\frac{x}{2a}\right)^{\frac{1}{2}}; \quad \frac{dy}{ds} = \left(\frac{2a-x}{2a}\right)^{\frac{1}{2}};$$

$$R_x = 2x^{\frac{1}{2}}(2a-x)^{\frac{1}{2}}; \quad R_y = -(2x-3a);$$

$$R = 4a\left(\frac{2a-x}{2a}\right)^{\frac{1}{2}}$$

Hence,

$$R_x \frac{dx}{ds} + R_y \frac{dy}{ds} = 3a\left(\frac{2a-x}{2a}\right)^{\frac{1}{2}}.$$

$$\therefore\ R = \frac{4}{3}\left(R_x \frac{dx}{ds} + R_y \frac{dy}{ds}\right).$$

When $R_x = 0,\ x = 0;$ or $x = 2a.$

When $R_y = 0,\ x = \frac{3}{2}a.$

These values, indicating the position of the cusps in the two curves, agree with the known characteristics of the caustics derived from the cycloid.

(IX.) It is required to determine the locus of the vertex and focus of a variable parabola which has contact of the second order with a given curve; its axis being parallel to that of the abscissæ.

Let X and Y be co-ordinates of the parabola which is in contact with a given curve $f(x\,y) = 0$, ξ and η co-ordinates of its vertex; the equation of the parabola will be,

$$(Y - \eta)^2 = 4m\,(X - \xi): \qquad (1)$$

$4m$ being the latus rectum of the parabola.

Writing u and v for the first and second differential co-efficients of the curve, as in article (II.), and assuming these co-efficients, as well as the co-ordinates, identical with those of the parabola; we shall find, after two differentiations of equation (1),

$$\xi = x + \frac{u}{2v};$$

$$m = -\frac{u^3}{2v}.$$

$$\eta = y + \frac{u^2}{v};$$

The elimination of x and y from the two first of these expressions and the given equation will determine the locus of the vertex of the parabola.

It is to be observed that, if X is less than ξ, *i.e.* if the vertex of the osculator is beyond the point of contact, m must be taken with the positive sign.

In ascertaining the locus of the foci, these expressions will remain the same with the exception of ξ, which must be corrected by the focal distance m;

$$\therefore \quad \xi_x = x + \frac{u}{2v}(1 - u^2); \quad \eta_x = y + \frac{u^2}{v};$$

and these are the usual formulæ for the co-ordinates of the caustic due to parallel incident rays, as determined by an independent method.

(X.) If the axis of the osculating parabola is parallel to the ordinates, its equation will be,

$$(X - \xi)^2 = 4m\,(Y - \eta); \qquad (2)$$

and we may either proceed as before, or, supposing the differentiation taken with respect to y, and replacing $u\ v$ by $u_{\prime}\ v_{\prime}$, the formulæ already obtained will give, by interchanging $\xi\ \eta$, $x\ y$,

$$\xi = x + \frac{u_{\prime}^2}{v_{\prime}}; \quad \eta = y + \frac{u_{\prime}}{2v_{\prime}}; \quad m = -\frac{u_{\prime}^3}{2v_{\prime}}.$$

By changing the independent variable from y to x, so that,

$$u_{\prime} = \frac{1}{u}; \quad v_{\prime} = -\frac{v}{u^3};$$

we find as co-ordinates of the vertex,

$$\xi = x - \frac{u}{v}; \qquad m = \frac{1}{2v}.$$

$$\eta = y - \frac{u^2}{2v};$$

In ascertaining the co-ordinates of the focus, ξ will now remain unchanged, while η must be corrected by the value of m; hence we obtain in relation to the focus,

$$\xi_y = x - \frac{u}{v}; \quad \eta_y = y + \frac{1}{2v}(1 - u^2).$$

In the application of these formulæ, if in any case Y is less than η, m must be taken with the negative sign.

(XI.) The determination of the caustic in this manner presents nothing new in principle; it may, however, be remarked that it involves a curious and striking analogy between the caustic and the evolute, in respect to the mode of their derivation from a given reflector. In fact we observe that, as the variable parabola which has been here considered, has contact of the *second* order with the reflector, so that the first and second differential co-efficients are identical in both for the point of osculation, the evolute of the reflector and that of the parabola of contact will be touched in the same point by the common normal. From this consideration may be inferred the general property that, *the evolute of any curve is the envelope of the evolutes of the parabola which has contact with it of the second order.* This conclusion, it is evident, would be equally true of any other curve having contact of the second order with the primitive; but it is of peculiar interest in the case of the parabola, for the intimate relation thus established between the evolute of any curve, and that of its caustic due to parallel incident rays; that is to say, that while the *focus* of the variable parabola traces the caustic of the reflector, the intersection of its consecutive *evolutes* will generate the evolute of that curve.

Cor. 1. From what has been here stated it will follow, that the evolute of the reflector is touched at the same point by each of the semi-cubical parabolas which are the evolutes of the parabolas of osculation; from which it may be concluded that, *the evolute of any curve is the envelope of three different systems of lines:* that is to say, the normal of the curve, and the evolutes of two parabolas having contact with it of the second order, whose axes intersect at right angles.

Cor. 2. Now since *any* curve which has contact of the second order with a given curve, will have, at the point of contact, a common normal with it, and since an indefinite number of curves may be assigned having contact of the second order at least, with any given curve, it follows that the evolute of any curve whatever is the envelope of these of an indefinite number of curves having contact of this order with the primitive. Further, since every curve is the evolute of its *involute*, whether or not this can be analytically defined, we must conclude that any curve may be regarded as the enveloping line of an infinite number of other curves of which the involutes have contact of the second order with the involute of the primitive.

(XII.) In illustration of the formulæ which have been investigated in articles (IX.) and (X.), we proceed to ascertain the locus of the foci of parabolas which have contact of the second order with a given Ellipse, defined by the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

In this case we find,

$$u = -\frac{b^2x}{a^2y}; \quad v = -\frac{b^4}{a^2y^3};$$

and, first, let the variable parabola have its axis parallel to that of x, then the formulæ of (IX.) will give,

$$\xi = x + \frac{u}{2v}(1 - u^2); \quad \eta = y + \frac{u^2}{v};$$

consequently, $$y = b^{\frac{2}{3}} \eta^{\frac{1}{3}}:$$

and, $$\xi = \frac{x}{2a^2}\left\{(2a^2 - b^2) + (a^2 + b^2)\left(\frac{\eta}{b}\right)^{\frac{2}{3}}\right\}.$$

$$\therefore \ x = \frac{2a^2\xi}{(2a^2 - b^2) + (a^2 + b^2)\left(\frac{\eta}{b}\right)^{\frac{2}{3}}}.$$

By eliminating x and y from (1), there is,

$$\xi = \pm \frac{a^2 + b^2}{2a}\left\{\frac{2a^2 - b^3}{a^2 + b^2} + \frac{\eta^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right\}\left(1 - \frac{\eta^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right)^{\frac{1}{2}}; \qquad (2)$$

representing the locus of the foci of the osculating parabolas: that is, the caustic of the ellipse touched by rays incident parallel to its major axis.

Secondly, let the axis of the osculating parabola be parallel to that of y; then, from the formulæ in (X.),

$$\xi = x - \frac{u}{v}; \quad \eta = y + \frac{1}{2v}(1 - u^2);$$

and the reduction of these expressions will give,

$$x = a^{\frac{2}{3}} \xi^{\frac{1}{3}}:$$

$$y = \frac{2b^2\eta}{(2b^2 - a^2) + (a^2 + b^2)\frac{\xi^{\frac{2}{3}}}{a^{\frac{2}{3}}}}$$

By elimination of x and y from (1) we find,

$$\eta = \pm \frac{a^2 + b^2}{2b} \left\{ \frac{2b^2 - a^2}{a^2 + b^2} + \frac{\xi^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right\} \left(1 - \frac{\xi^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right)^{\frac{1}{2}}; \quad (3)$$

representing the locus of the foci; that is, the caustic of the ellipse touched by rays incident parallel to its minor axis.

The curves represented by (2) and (3) are similar in character, and are reciprocally convertible, as they necessarily should be, by interchanging the axes and co-ordinates.

COR. 1. Since the ellipse has been referred to its centre, by writing $b\sqrt{-1}$ for b we shall obtain, without a separate investigation, the analogous caustics when the reflector is an hyperbola; their equations then will be, when rays are incident parallel,

1st. to the *major* axis,

$$\xi = \pm \frac{a^2 - b^2}{2a} \left\{ \frac{2a^2 + b^2}{a^2 - b^2} - \frac{\eta^{\frac{2}{3}}}{b^{\frac{2}{3}}} \right\} \left(1 + \frac{\eta^{\frac{2}{3}}}{b^{\frac{2}{3}}}\right)^{\frac{1}{2}} \quad (4)$$

2nd. to the *minor* axis,

$$\eta = \pm \frac{a^2 - b^2}{2b} \left\{ \frac{2b^2 + a^2}{a^2 - b^2} - \frac{\xi^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right\} \left(\frac{\xi^{\frac{2}{3}}}{a^{\frac{2}{3}}} - 1\right)^{\frac{1}{2}}. \quad (5)$$

COR. 2. When the reflector is a *circle* the equation (3) becomes,

$$\eta = \pm \frac{1}{2}\left(a^{\frac{2}{3}} + 2\xi^{\frac{2}{3}}\right)\left(a^{\frac{2}{3}} - \xi^{\frac{2}{3}}\right)^{\frac{1}{2}},$$

representing the caustic epicycloid of two cusps.

Again. if the reflecting *hyperbola* is rectangular, the equations (4) and (5) are simplified to the following forms:

$$\xi = \pm \frac{3a}{2}\sqrt{\left(\frac{\eta}{a}\right)^{\frac{2}{3}} + 1}. \qquad \eta = \pm \frac{3a}{2}\sqrt{\left(\frac{\xi}{a}\right)^{\frac{2}{3}} - 1}.$$

COR. 3. When the reflector is elliptic, the caustic, in general, will be a curve whose characteristics resemble closely those of the epicycloid, and it may be delineated without difficulty from the equations (2) and (3). There is, however, one case in which these equations become greatly simplified. Let the ellipse be one in which $2b^2 = a^2$, then the equation of the caustic when rays are incident parallel to the minor axis will be,

$$\eta = \pm \frac{3}{2} \xi^{\frac{2}{3}} \sqrt{\tfrac{1}{2} (a^{\frac{2}{3}} - \xi^{\frac{2}{3}})}; \tag{6}$$

while, for the co-related caustic, the equation becomes,

$$\xi = \pm \frac{3a}{4b} (b^{\frac{2}{3}} + \eta^{\frac{2}{3}}) \sqrt{b^{\frac{2}{3}} - \eta^{\frac{2}{3}}}. \tag{7}$$

In figure (2) the co-related caustics (6) and (7) are drawn, together with the evolute of the reflecting ellipse. The caustic due to rays incident parallel to the major axis has its cusps on that axis at distances $= \pm \frac{3}{4} a$ from the centre, and is in contact with the ellipse at the extremities of its minor axis. The caustic which is generated by rays incident parallel to the minor axis is in contact with the ellipse at the extremities of its major axis, and has two osculating cusps at the centre.

(XIII.) As an example of the theorem in (XI.) we may take the case of the circle, $x^2 + y^2 = a^2$; the conclusion drawn will then require that the evolutes of all its osculating parabolas should pass through the centre.

1st. Let the axis of the parabola be parallel to the transverse diameter; ξ_x and η_x co-ordinates of its vertex. Then,

$$\xi_x = x + \frac{u}{2v};\ \eta_x = y + \frac{u^2}{v};\ m = \frac{u^3}{2v}$$

and,

$$u = -\frac{x}{y};\ v = -\frac{a^2}{y^3}$$

$$\therefore\ \xi_x = \frac{x\,(3a^2 - x^2)}{2a^2};\ \eta_x = \frac{y^3}{a^2};\ m = \frac{x^3}{2a^2}$$

Let now $\xi_1\ \eta_1$ be co-ordinates of an evolute, measured from the vertex of the variable parabola; it may be readily shewn that its equation is,

$$m\left(\frac{\eta_1}{2}\right)^2 = \left(\frac{\xi_1 - 2m}{3}\right)^3 : \qquad (1)$$

and, if ξ η are co-ordinates of the evolute measured from the centre of the circle,

$$\xi_1 - 2m = \xi_x - \xi - 2m = \frac{3xy^2}{2a^2} - \xi.$$

$$\eta_1 = \eta_x - \eta = \frac{y^3}{a^2} - \eta.$$

After substituting in (1), the equation of the variable evolute is found to be,

$$27\, x^3\, (y^3 - a^2\eta)^2 = (3xy^2 - 2a^2\xi)^3. \qquad (2)$$

On giving to x a variation in (2) there will result,

$$\frac{(3xy^2 - 2a^2\xi)^2}{9x^2\,(y^3 - a^2\eta)} = \frac{(y^2 - 2x^2)\,y - a^2\eta}{y^2 - 2x^2}. \qquad (3)$$

The elimination of x y between (2) and (3) combined with the equation of the circle, will give the envelope equation; but it is sufficient for the point in view to observe that, since (2) and (3) are satisfied by the condition $\xi = o$, $\eta = o$, one branch of the varying evolute always passes through the centre.

2nd. Let the axis of the osculator be parallel to the ordinates. It is obvious that it is only necessary to interchange the co-ordinates in the equations already determined; or the calculation may be performed independently by means of the formulæ in (X.). We find, then, as the equation of any evolute,

$$27\, y^3\, (x^3 - a^2\xi)^2 = (3x^2y - 2a^2\eta)^3;$$

and, the derived equation is,

$$\frac{(3x^2y - 2a^2\eta)^2}{9y^2\,(x^3 - a^2\xi)} = \frac{(x^2 - 2y^2)\,x - a^2\xi}{x^2 - 2y^2}:$$

both of which are satisfied by the same assumption as before. Consequently a branch of each of the parabolic evolutes passes through the centre, which is the evanescent evolute of the circle.

(XIV.) In defining the Evolute of the Lemniscata by an equation expressed in terms of rectangular co-ordinates, the complexity of the auxiliary equations which would be ordinarily used may be dispensed with, by determining the virtual caustic by reflection due to the hyperbola, when rays reflected from its arc emerge from the centre of that curve; regarded as a centre of light. It is obvious that this inverse process, as it may in one sense be termed, is applicable in ascertaining the evolutes of other curves, the case of the lemniscata being here selected chiefly because, in the knowledge of the writer, its evolute has not previously received any analytical expression. Such an investigation may, perhaps, be further considered to be worth some degree of attention, not so much in annexing an isolated instance to a class of curves which present but few complete and satisfactory results to the analyst, as by illustrating a method of deduction which does not appear to have been hitherto applied to problems of this nature.

Let the hyperbolic reflector be rectangular, referred to its centre and principal axes (fig. 3), x and y co-ordinates of any point P on the curve, Pt and PG the tangent and normal lines, θ the inclination of an incident ray O P to the axis of x; we have then the relations,

$$x^2 - y^2 = a^2; \quad \frac{dy}{dx} = \frac{x}{y} = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right);$$

$$\therefore\ \mathrm{P}tx = \frac{\pi}{2} - \theta; \text{ and } \mathrm{PGO} = \theta.$$

Let the ray OP, after reflection at P, meet the axis of x in the point R, i being the angle of incidence; then,

$$\mathrm{PRG} = \mathrm{RP}t + \mathrm{P}tx = \left(\frac{\pi}{2} - i\right) + \left(\frac{\pi}{2} - \theta\right) = \pi - \theta - i,$$

$$\text{but, } i = 2\theta; \quad \therefore\ \mathrm{PRG} = \pi - 3\theta.$$

If, now, ξ and η are co-ordinates of the reflected ray, its equation will be,

$$\frac{\eta - y}{\xi - x} = \tan \mathrm{PRG} = -\tan 3\theta = -\frac{y\,(3x^2 - y^2)}{x\,(x^2 - 3y^2)};$$

which may be written in the form,

$$\eta + \frac{y\,(3x^2 - y^2)}{x\,(x^2 - 3y^2)}\,\xi = \frac{4y\,(x^2 - y^2)}{x^2 - 3y^2}. \qquad (1)$$

(XV.) On giving to the parameters x and y a variation in the equation (1) we shall find, after the requisite eliminations,

$$\xi = \frac{4x^3}{3\,(x^2 + y^2)}; \quad \eta = \frac{4y^3}{3\,(x^2 + y^2)}. \qquad (2)$$

The relations (2) will give the condition $\frac{\xi}{\eta} = \left(\frac{x}{y}\right)^3$, which is equivalent to $\cot\psi = \cot{}^3\theta$; ψ being the angle at which OQ, the radius vector of the caustic, is inclined to the axis of the reflector. We have from this condition,

$$y = \left(\frac{\eta}{\xi}\right)^{\frac{1}{3}} x,$$

from which (2) will give the expressions,

$$x = \frac{3}{4}\,\xi^{\frac{1}{3}}\,(\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}});\quad y = \frac{3}{4}\,\eta^{\frac{1}{3}}(\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}});$$

and, after elimination of x and y from the equation of the reflector, $x^2 - y^2 = a^2$, there is found,

$$(\xi^{\frac{2}{3}} - \eta^{\frac{2}{3}})\ (\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}})^2 = \frac{16a^2}{9}, \qquad (3)$$

representing the curve touched by the reflected rays when produced through the reflector; which may be termed the virtual caustic.

(XVI.) On expanding the equation (3) we find for the asymptotes of the caustic,

$$1 - \left(\frac{\eta}{\xi}\right)^{\frac{4}{3}} = o;\quad \therefore\ \eta = \pm\,\xi.$$

There is, also, $$\frac{d\eta}{d\xi} = \left(\frac{\eta}{\xi}\right)^{\frac{1}{3}} \frac{3\xi^{\frac{2}{3}} - \eta^{\frac{2}{3}}}{3\eta^{\frac{2}{3}} - \xi^{\frac{2}{3}}}. \qquad (4)$$

Hence, if $\eta = o$, $\frac{d\eta}{d\xi} = o$, and $\xi = \pm\,\frac{4a}{3}$, indicating two cusps on the axis of the reflector, at distances from its centre $= \pm\,\frac{4a}{3}$; and if $\eta = \pm\,\xi$, $\frac{d\eta}{d\xi} = \pm\,1$, corresponding to the asymptotes.

Again, if $\frac{d\eta}{d\xi} = \infty$, we shall find,

$$\xi = \pm\,\frac{a}{2}\sqrt{6};\quad \eta = \pm\,\frac{a}{6}\sqrt{2};$$

for the points at which the tangent of the caustic cuts its axis at right angles: and it is evident that these points correspond with *minimum* values of ξ, as may be independently shewn from (2).

It appears, therefore, that this caustic consists of four infinitely extended branches which spring from cusps on the axis of the reflector at the distances $\pm\,\frac{4a}{3}$ from its centre; these branches approaching asymptotes which coincide with those of the reflector.

(XVII.) Since the lemniscata is the orthogonal trajectory, or first curve of equiangular intersection, derived from the hyperbola, the caustic of the hyperbola whose axis $= 2a$ is the evolute of the lemniscata whose axis $= 4a$;* consequently, the lemniscata given by the equation

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2),$$

has for the equation of its evolute,

$$(\xi^{\frac{2}{3}} - \eta^{\frac{2}{3}}) (\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}})^2 = \frac{4a^2}{9}; \qquad (5)$$

which represents a curve similar to the caustic, springing from cusps at the distances $\xi = \pm \frac{2}{3}a$, and approaching the same asymptotes.

(XVIII.) The character of the evolute may be otherwise, and, perhaps, the most readily ascertained by reference to the linear-polar co-ordinates.

Let r and p be the radius vector of the lemniscata and the perpendicular on its tangent, ρ and q the corresponding co-ordinates of the evolute; then we shall find, $a^2p = r^3$:

$$q^2 = p^{\frac{2}{3}} (a^{\frac{4}{3}} - p^{\frac{4}{3}}) : \quad (\rho^2 - q^2)^{\frac{1}{2}} = \frac{a}{3}\left(\frac{a}{p}\right)^{\frac{1}{3}} - p.$$

By the elimination of p from these equations may be obtained, as the equation of the evolute,

$$\left(\frac{v}{a}\right)^4 + \left(\frac{q}{v}\right)^2 = 1; \qquad (6)$$

in which, $v^2 = \frac{3}{2}\left\{\rho^2 \pm \left(\rho^4 - \frac{4a^4}{27}\right)^{\frac{1}{2}}\right\}$.

The equation (6) may be rationalised without difficulty, but as its form in any case is complicated, it is better to make ρ and q each a function of one of the co-ordinates of the primitive, as r; then,

$$q = \frac{r}{a^2}(a^4 - r^4)^{\frac{1}{2}}; \qquad \rho = \frac{1}{3r}(3r^4 + a^4)^{\frac{1}{2}};$$

and, if R and $R_,$ are the radii of curvature of the primitive and its evolute, there is,

$$R = \frac{a^2}{3r} = \frac{a^{\frac{4}{3}}}{3p^{\frac{1}{3}}}; \quad R_, = q\frac{dR}{dp} = -\frac{a^{\frac{4}{3}}}{9p}\left(a^{\frac{4}{3}} - p^{\frac{4}{3}}\right)^{\frac{1}{2}}.$$

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* See Ray Surfaces of Reflection, page 124, by the writer of this paper. Longman & Co., 1857.

When $r = a = p$, $R_{,} = o$, and $q = o$; indicating a cusp on the axis when $\rho = \frac{2}{3}a$. Again, when $r = o$, $R = \infty$, $q = o$; while $\rho = \infty$ and $R_{,} = \infty$; conditions which determine the asymptotes of the evolute, corresponding with the inflections of the lemniscata.

(XIX.) The evolute of this curve has the curious property that if values of r are taken *decreasing* from its maximum in any geometric series whose common ratio is $\frac{1}{k}$, the corresponding arcs of the evolute, commencing from a cusp, will be always the terms of an *increasing* geometric progression of which the common ratio is k.

Let S be an arc of the evolute, measured from a cusp; then, since $R = \frac{a^2}{3r}, = \frac{a}{3}$ when $r = a$, we shall have,

$$S = \frac{a}{3}\left(\frac{a}{r} - 1\right).$$

Now, let $\quad r = \frac{a}{k^n}, \quad \therefore \quad S = (k^n - 1)\frac{a}{3};$

and let, $\quad S_0; \quad S_1; \quad S_2; \quad \ldots \quad S_n;$

$$o; \quad (k-1)\frac{a}{3}; \quad (k^2-1)\frac{a}{3}; \quad \ldots \quad (k^n-1)\frac{a}{3};$$

be the values assumed by S corresponding to the consecutive values of n.

By taking the difference between consecutive terms in the preceding series, we find generally,

$$S_n - S_{n-1} = k^{n-1}S_1$$

But $S_n - S_{n-1}$ is the arc of the evolute intercepted between the radii of curvature at the points on the curve, $\frac{a}{k^n}$, $\frac{a}{k^{n-1}}$; and we have, consequently, the following series, expressing the successive values of the radii of the lemniscata and the arcs of its evolute which correspond to consecutive points taken on the curve, according to the law before expressed; viz.:—

$$(r = \;) \quad a; \quad \frac{a}{k}; \quad \frac{a}{k^2}; \quad \ldots \quad \frac{a}{k^n}.$$

$$(S = \;) \quad o; \quad S_{,}; \quad kS_{,}; \quad \ldots \quad k^{n-1}S_{,}.$$

RÉSUMÉ OF PROPOSITIONS.

In the following brief summary are included under one view some of the principal results, in relation to ray-surfaces and caustics of reflection, which have been obtained by the author of this tract: and which appear for the most part to have been heretofore unknown, or unrecorded.

(I.) RAY-SURFACES OF REFLECTION.

(1.) General equations for determining the ray-surface, when rays, divergent or parallel, are incident upon any plane reflector.

(2.) The caustic *in plano*, due to parallel rays is the line of intersection between the cylindrical surface-caustic and the plane of the reflector.

(3.) Discussion of the ray-surface under different circumstances, when the reflector is circular.

(4.) The envelope of ray-surfaces generated by parallel rays, incident at all degrees of elevation on a circular reflector, is an epicycloïdal cylinder.

(5.) Verification of the theorem in (2), the reflector circular.

(6.) Discussion of the enveloping surface of the ray-surfaces when rays emanating from a luminous spheröid are reflected from a circular ring. Application of the result to the optical system of a ringed planet.

(7.) Discussion of the cuneo-trochöid; being the ray-surface due to parallel incident rays reflected from the cycloid.
Geometrical constructions for the trochöidal sections.
Remarkable line of no aberration in this surface.

(8.) Verification of the theorem in (2.); the reflector a cycloid.

(9.) Determination of the ray-surface due to parallel rays incident on a parabolic reflector.

(10.) Accurate reflection to the focus deduced as a corollary from general parabolic reflection.

(11.) Verification of the theorem in (2.); the reflector parabolic.

(12.) Determination of the general equation of ray-surfaces generated by vertical reflection from any plane algebraic curve of the nth degree.

(II.) APPENDIX TO RAY-SURFACES.

(1.) Properties of the orthogonal trajectory, or first curve of equiangular intersection, as connected with caustics generated by divergent rays.

(2.) Differential equations for the evolute and involute of a plane curve; and for the nth evolute; in polar co-ordinates.

(3.) Radius of curvature of the orthogonal trajectory always an arithmetic mean between the radius-vector of the reflector and the tangent of its caustic due to divergent rays.

(4.) New formula for the determination of an asymptotic circle.

(5.) Involute of the parabola, and transcendental series for its arc. Derivation from this series of the remarkable trigonometrical equivalence,

$$\frac{\sin\psi}{1^2} - \frac{\sin 3\psi}{3^2} + \frac{\sin 5\psi}{5^2} - \ldots = \frac{\pi}{4}\psi.^{*}$$

(6.) Curve whose radius-vector is equal to that of its evolute. This curve includes the logarithmic spiral, and has a similar property of reproduction.

(7.) Caustics derived from the following curves, viz.: Spiral of Archimedes. Involute of the Circle. Hyperbolic Spiral. Parabola, rays emanating from its vertex. Evolute of the parabola; the caustic is the evolute of another parabola. Cissöid. Cardiöide. Lemniscata.

(8.) Converse propositions.—Investigation of the Reflector whose caustic is a Circle.—Reflector whose caustic is an Epicycloid or Hypocycloid.—Reflector whose caustic is the evolute of an ellipse.

(III.) RELATED CAUSTICS.

(1.) Equation connecting the radius of curvature of any reflector with those of two related caustics. Geometrically, the radius of the reflector = the sum of the caustic radii projected on its *normal*.

(2.) The radius of curvature of the reflector = twice the distance between corresponding points in the related caustics: with other theorems connecting the three related curves.

(3.) Equation connecting the arc of the reflector with those of two related caustics. Geometrically, the elementary arc of any reflector = the sum of the elementary arcs of the related caustics projected on the *tangent*.

(4.) Relation between the Evolute of the Reflector and its Caustics, deduced from the osculating parabola. While the focus traces the caustic, the evolute of the reflector is generated by the intersection of the evolutes of two systems of osculating parabolas. Verification of this when the reflector is circular. General inference respecting the analytical character of every plane curve.

(5.) Equation to the evolute of a lemniscata deduced from the caustic of the hyperbola.

(6.) Singular property of the arc of this evolute.

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* See De Morgan's Differential and Integral Calculus, p. 608, Definite Integrals; where the series is established, of which this is the integral.

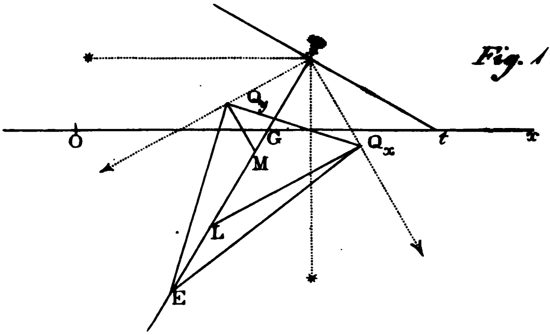

Fig. 1.

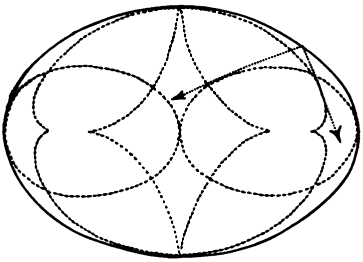

Fig. 2.

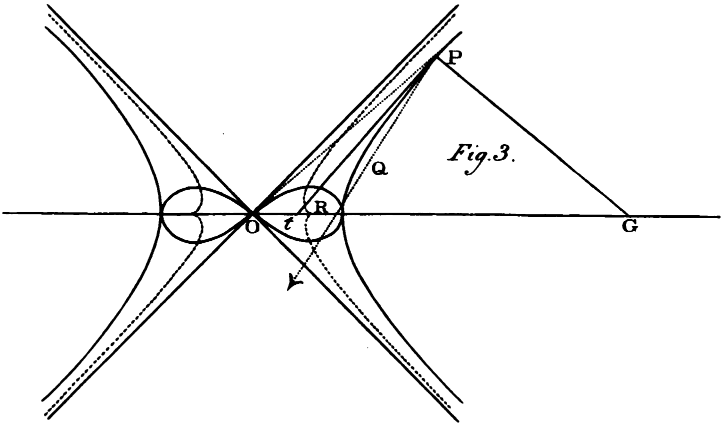

Fig. 3.